Let E be a set of universally quantified equations. A model $\mathcal{A}$ of E is also called an *E-algebra*. If $E \models \forall \vec{x}(s \approx t)$, i.e., $\forall \vec{x}(s \approx t)$ is valid in all E-algebras, this is also denoted with $s \approx_E t$. The goal is to use the rewrite relation $\rightarrow_E$ to express the semantic consequence relation syntactically: $s \approx_E t$ if and only if $s \leftrightarrow_E^* t$.

Let E be a set of (well-sorted) equations over $T(\Sigma, \mathcal{X})$ where all variables are implicitly universally quantified. The following inference system allows to derive consequences of E:

Reflexivity $E \Rightarrow_{\mathrm{E}} E \cup \{t \approx t\}$

Symmetry $E \uplus \{t \approx t'\} \Rightarrow_{\mathrm{E}} E \cup \{t \approx t'\} \cup \{t' \approx t\}$

Transitivity $E \uplus \{t \approx t', t' \approx t''\} \Rightarrow_{\mathrm{E}}$
$E \cup \{t \approx t', t' \approx t''\} \cup \{t \approx t''\}$

Congruence $E \uplus \{t_1 \approx t'_1, \ldots, t_n \approx t'_n\} \ \Rightarrow_{\mathrm{E}}$
$E \cup \{t_1 \approx t'_1, \ldots, t_n \approx t'_n\} \cup \{f(t_1, \ldots, t_n) \approx f(t'_1, \ldots, t'_n)\}$
for any function $f : \mathrm{sort}(t_1) \times \ldots \times \mathrm{sort}(t_n) \to S$ for some S

Instance $E \uplus \{t \approx t'\} \ \Rightarrow_{\mathrm{E}} \ E \cup \{t \approx t'\} \cup \{t\sigma \approx t'\sigma\}$
for any well-sorted substitution σ

4.1.5 Lemma (Equivalence of $\leftrightarrow_E^*$ and $\Rightarrow_E^*$)

The following properties are equivalent:

1. $s \leftrightarrow_E^* t$
2. $E \Rightarrow_E^* s \approx t$ is derivable.

where $E \Rightarrow_E^* s \approx t$ is an abbreviation for $E \Rightarrow_E^* E'$ and $s \approx t \in E'$.

4.1.6 Corollary (Convergence of E)

If a set of equations E is convergent then $s \approx_E t$ if and only if $s \leftrightarrow^* t$ if and only if $s\downarrow_E = t\downarrow_E$.

4.1.7 Corollary (Decidability of $\approx_E$)

If a set of equations E is finite and convergent then $\approx_E$ is decidable.

The above Lemma 4.1.5 shows equivalence of the syntactically defined relations $\leftrightarrow_E^*$ and *Rightarrow*$_E^*$. What is missing, in analogy to Herbrand's theorem for first-order logic without equality Theorem 3.5.5, is a semantic characterization of the relations by a particular algebra.

4.1.8 Definition (Quotient Algebra)

For sets of unit equations this is a *quotient algebra*: Let X be a set of variables. For $t \in T(\Sigma, \mathcal{X})$ let $[t] = \{t' \in T(\Sigma, \mathcal{X})) \mid E \Rightarrow_{\mathrm{E}}^* t \approx t'\}$ be the *congruence class* of t. Define a Σ-algebra $\mathcal{I}_E$, called the *quotient algebra*, technically $T(\Sigma, \mathcal{X})/E$, as follows: $S^{\mathcal{I}_E} = \{[t] \mid t \in T_S(\Sigma, \mathcal{X})\}$ for all sorts S and $f^{\mathcal{I}_E}([t_1], \ldots, [t_n]) = [f(t_1, \ldots, t_n)]$ for $f : \mathrm{sort}(t_1) \times \ldots \times \mathrm{sort}(t_n) \to T \in \Omega$ for some sort T.

4.1.9 Lemma ($\mathcal{I}_E$ is an E-algebra)

$\mathcal{I}_E = T(\Sigma, \mathcal{X})/E$ is an E-algebra.

4.1.10 Lemma ($\Rightarrow_E$ is complete)

Let $\mathcal{X}$ be a countably infinite set of variables; let $s, t \in T_S(\Sigma, \mathcal{X})$.
If $\mathcal{I}_E \models \forall \vec{x}(s \approx t)$, then $E \Rightarrow_E^* s \approx t$ is derivable.

4.1.11 Theorem (Birkhoff's Theorem)

Let $\mathcal{X}$ be a countably infinite set of variables, let E be a set of (universally quantified) equations. Then the following properties are equivalent for all $s, t \in T_S(\Sigma, \mathcal{X})$:

1. $s \leftrightarrow_E^* t$.
2. $E \Rightarrow_E^* s \approx t$ is derivable.
3. $s \approx_E t$, i.e., $E \models \forall \vec{x}(s \approx t)$.
4. $\mathcal{I}_E \models \forall \vec{x}(s \approx t)$.

By Theorem 4.1.11 the semantics of E and $\leftrightarrow_E^*$ conincide. In order to decide $\leftrightarrow_E^*$ we need to turn $\rightarrow_E^*$ in a confluent and terminating relation.

If $\leftrightarrow_E^*$ is terminating then confluence is equivalent to local confluence, see Newman's Lemma, Lemma 1.6.6. Local confluence is the following problem for TRS: if $t_1 \; {}_E\!\leftarrow t_0 \rightarrow_E t_2$, does there exist a term s so that $t_1 \rightarrow_E^* s \; {}_E^*\!\leftarrow t_2$?

If the two rewrite steps happen in different subtrees (disjoint redexes) then a repitition of the respective other step yields the common term s.

If the two rewrite steps happen below each other (overlap at or below a variable position) again a repetition of the respective other step yields the common term s.

If the left-hand sides of the two rules overlap at a non-variable position there is no ovious way to generate s.

More technically two rewrite rules $l_1 \to r_1$ and $l_2 \to r_2$ overlap if there exist some non-variable subterm $l_1|_p$ such that l_2 and $l_1|_p$ have a common instance $(l_1|_p)\sigma_1 = l_2\sigma_2$. If the two rewrite rules do not have common variables, then only a single substitution is necessary, the mgu σ of $(l_1|_p)$ and l_2.

4.2.1 Definition (Critical Pair)

Let $l_i \to r_i$ ($i = 1, 2$) be two rewrite rules in a TRS R whithout common variables, i.e., $\text{vars}(l_1) \cap \text{vars}(l_2) = \emptyset$. Let $p \in \text{pos}(l_1)$ be a position so that $l_1|_p$ is not a variable and σ is an mgu of $l_1|_p$ and l_2. Then $r_1\sigma \leftarrow l_1\sigma \to (l_1\sigma)[r_2\sigma]_p$.

$\langle r_1\sigma, (l_1\sigma)[r_2\sigma]_p\rangle$ is called a *critical pair* of R.

The critical pair is *joinable* (or: converges), if $r_1\sigma \downarrow_R (l_1\sigma)[r_2\sigma]_p$.

4.2.2 Theorem (“Critical Pair Theorem”)

A TRS R is locally confluent iff all its critical pairs are joinable.

Knuth-Bendix Completion (KBC)

Given a set E of equations, the goal of Knuth-Bendix completion is to transform E into an equivalent convergent set R of rewrite rules. If R is finite this yields a decision procedure for E.

For ensuring termination the calculus fixes a reduction ordering $\succ$ and constructs R in such a way that $\rightarrow_R \subseteq \succ$, i.e., $l \succ r$ for every $l \rightarrow r \in R$.

For ensuring confluence the calculus checks whether all critical pairs are joinable.

The completion procedure itself is presented as a set of abstract rewrite rules working on a pair of equations E and rules R: $(E_0;R_0) \Rightarrow_{\mathrm{KBC}} (E_1;R_1) \Rightarrow_{\mathrm{KBC}} (E_1;R_2) \Rightarrow_{\mathrm{KBC}} \ldots$.

The initial state is $(E_0, \emptyset)$ where $E = E_0$ contains the input equations.

If $\Rightarrow_{\mathrm{KBC}}$ successfully terminates then E is empty and R is the convergent rewrite system for E_0.

For each step $(E; R) \Rightarrow_{\mathrm{KBC}} (E'; R')$ the equational theories of $E \cup R$ and $E' \cup R'$ agree: $\approx_{E \cup R} = \approx_{E' \cup R'}$. By $\mathrm{cp}(R)$ I denote the set of critical pairs between rules in R.

Orient $(E \uplus \{s \mathrel{\dot{\approx}} t\}; R) \;\Rightarrow_{\mathrm{KBC}}\; (E; R \cup \{s \to t\})$
if $s \succ t$

Delete $(E \uplus \{s \approx s\}; R) \;\Rightarrow_{\mathrm{KBC}}\; (E; R)$

Deduce $(E; R) \;\Rightarrow_{\mathrm{KBC}}\; (E \cup \{s \approx t\}; R)$
if $\langle s, t\rangle \in \mathrm{cp}(R)$

Simplify-Eq $(E \uplus \{s \dot{\approx} t\}; R) \Rightarrow_{\text{KBC}} (E \cup \{u \approx t\}; R)$

if $s \rightarrow_R u$

R-Simplify-Rule $(E; R \uplus \{s \rightarrow t\}) \Rightarrow_{\text{KBC}} (E; R \cup \{s \rightarrow u\})$

if $t \rightarrow_R u$

L-Simplify-Rule $(E; R \uplus \{s \rightarrow t\}) \Rightarrow_{\text{KBC}} (E \cup \{u \approx t\}; R)$

if $s \rightarrow_R u$ using a rule $l \rightarrow r \in R$ so that $s \sqsupset l$, see below.

Trivial equations cannot be oriented and since they are not needed they can be deleted by the Delete rule.

The rule Deduce turns critical pairs between rules in R into additional equations. Note that if $\langle s, t\rangle \in \mathrm{cp}(R)$ then $s \,{}_R{\leftarrow}\, u \rightarrow_R t$ and hence $R \models s \approx t$.

The simplification rules are not needed but serve as reduction rules, removing redundancy from the state. Simplification of the left-hand side may influence orientability and orientation of the result. Therefore, it yields an equation. For technical reasons, the left-hand side of $s \rightarrow t$ may only be simplified using a rule $l \rightarrow r$, if $l \rightarrow r$ cannot be simplified using $s \rightarrow t$, that is, if $s \sqsupset l$, where the *encompassment quasi-ordering* $\gtrsim$ is defined by $s \gtrsim l$ if $s|_p = l\sigma$ for some p and σ and $\sqsupset = \gtrsim \setminus \lesssim$ is the strict part of $\gtrsim$.