# First-Order Ground Superposition

Propositional clauses and ground clauses are essentially the same, as long as equational atoms are not considered. This section deals only with ground clauses and recalls mostly the material from Section 2.7 for first-order ground clauses. The main difference is that the atom ordering is more complicated, see Section 3.11.

From now on let $N$ be a possibly infinite set of ground clauses.

## 3.12.1 Definition (Ground Clause Ordering)

Let $\prec$ be a strict rewrite ordering total on ground terms and ground atoms. Then $\prec$ can be lifted to a total ordering $\prec_L$ on literals by its multiset extension $\prec_{\mathrm{mul}}$ where a positive literal $P(t_1, \ldots, t_n)$ is mapped to the multiset $\{P(t_1, \ldots, t_n)\}$ and a negative literal $\neg P(t_1, \ldots, t_n)$ to the multiset $\{P(t_1, \ldots, t_n), P(t_1, \ldots, t_n)\}$.

The ordering $\prec_L$ is further lifted to a total ordering on clauses $\prec_C$ by considering the multiset extension of $\prec_L$ for clauses.

## 3.12.2 Proposition (Properties of the Ground Clause Ordering)

1. The orderings on literals and clauses are total and well-founded.
2. Let $C$ and $D$ be clauses with $P(t_1, \ldots, t_n) = \text{atom}(\max(C))$, $Q(s_1, \ldots, s_m) = \text{atom}(\max(D))$, where $\max(C)$ denotes the maximal literal in $C$.
   (a) If $Q(s_1, \ldots, s_m) \prec_L P(t_1, \ldots, t_n)$ then $D \prec_C C$.
   (b) If $P(t_1, \ldots, t_n) = Q(s_1, \ldots, s_m)$, $P(t_1, \ldots, t_n)$ occurs negatively in $C$ but only positively in $D$, then $D \prec_C C$.

Eventually, as I did for propositional logic, I overload $\prec$ with $\prec_L$ and $\prec_C$. So if $\prec$ is applied to literals it denotes $\prec_L$, if it is applied to clauses, it denotes $\prec_C$.

Note that $\prec$ is a total ordering on literals and clauses as well. For superposition, inferences are restricted to maximal literals with respect to $\prec$.

For a clause set $N$, I define $N^{\prec C} = \{D \in N \mid D \prec C\}$.

### 3.12.3 Definition (Abstract Redundancy)

A ground clause $C$ is *redundant* with respect to a set of ground clauses $N$ if $N^{\prec C} \models C$.

Tautologies are redundant. Subsumed clauses are redundant if $\subseteq$ is strict. Duplicate clauses are anyway eliminated quietly because the calculus operates on sets of clauses.

### 3.12.4 Definition (Selection Function)

The selection function sel maps clauses to one of its negative literals or $\bot$. If $\text{sel}(C) = \neg P(t_1, \ldots, t_n)$ then $\neg P(t_1, \ldots, t_n)$ is called *selected* in $C$. If $\text{sel}(C) = \bot$ then no literal in $C$ is *selected*.

The selection function is, in addition to the ordering, a further means to restrict superposition inferences. If a negative literal is selected in a clause, any superposition inference must be on the selected literal.

## 3.12.5 Definition (Partial Model Construction)

Given a clause set $N$ and an ordering $\prec$ we can construct a (partial) model $N_{\mathcal{I}}$ for $N$ inductively as follows:

$$
\begin{array}{rcl}
N_C & := & \bigcup_{D \prec C} \delta_D \\
\\
\delta_D & := & \left\{ \begin{array}{ll} \{P(t_1, \ldots, t_n)\} & \text{if } D = D' \vee P(t_1, \ldots, t_n), \\ & P(t_1, \ldots, t_n) \text{ strictly maximal, no literal} \\ & \text{selected in } D \text{ and } N_D \not\models D \\ \\ \emptyset & \text{otherwise} \end{array} \right. \\
N_{\mathcal{I}} & := & \bigcup_{C \in N} \delta_C
\end{array}
$$

Clauses $C$ with $\delta_C \neq \emptyset$ are called *productive*.

## 3.12.6 Proposition (Propertied of the Model Operator)

Some properties of the partial model construction.

1. For every $D$ with $(C \lor \neg P(t_1, \ldots, t_n)) \prec D$ we have $\delta_D \neq \{P(t_1, \ldots, t_n)\}$.
2. If $\delta_C = \{P(t_1, \ldots, t_n)\}$ then $N_C \cup \delta_C \models C$.
3. If $N_C \models D$ and $D \prec C$ then for all $C'$ with $C \prec C'$ we have $N_{C'} \models D$ and in particular $N_{\mathcal{I}} \models D$.
4. There is no clause $C$ with $P(t_1, \ldots, t_n) \lor P(t_1, \ldots, t_n) \prec C$ such that $\delta_C = \{P(t_1, \ldots, t_n)\}$.

Please properly distinguish: $N$ is a set of clauses interpreted as the conjunction of all clauses.

$N^{\prec C}$ is of set of clauses from $N$ strictly smaller than $C$ with respect to $\prec$.

$N_{\mathcal{I}}$, $N_C$ are Herbrand interpretations (see Proposition 3.5.3).

$N_{\mathcal{I}}$ is the overall (partial) model for $N$, whereas $N_C$ is generated from all clauses from $N$ strictly smaller than $C$.

**Superposition Left**
$(N \uplus \{C_1 \vee P(t_1,\ldots,t_n), C_2 \vee \neg P(t_1,\ldots,t_n)\}) \ \Rightarrow_{\text{SUP}}$
$(N \cup \{C_1 \vee P(t_1,\ldots,t_n), C_2 \vee \neg P(t_1,\ldots,t_n)\} \cup \{C_1 \vee C_2\})$
where (i) $P(t_1,\ldots,t_n)$ is strictly maximal in $C_1 \vee P(t_1,\ldots,t_n)$
(ii) no literal in $C_1 \vee P(t_1,\ldots,t_n)$ is selected
(iii) $\neg P(t_1,\ldots,t_n)$ is maximal and no literal selected in $C_2 \vee \neg P(t_1,\ldots,t_n)$, or $\neg P(t_1,\ldots,t_n)$ is selected in $C_2 \vee \neg P(t_1,\ldots,t_n)$

**Factoring** $(N \uplus \{C \vee P(t_1,\ldots,t_n) \vee P(t_1,\ldots,t_n)\}) \ \Rightarrow_{\text{SUP}}$
$(N \cup \{C \vee P(t_1,\ldots,t_n) \vee P(t_1,\ldots,t_n)\} \cup \{C \vee P(t_1,\ldots,t_n)\})$
where (i) $P(t_1,\ldots,t_n)$ is maximal in $C \vee P(t_1,\ldots,t_n) \vee P(t_1,\ldots,t_n)$
(ii) no literal is selected in $C \vee P(t_1,\ldots,t_n) \vee P(t_1,\ldots,t_n)$

### 3.12.7 Definition (Saturation)

A set $N$ of clauses is called *saturated up to redundancy*, if any inference from non-redundant clauses in $N$ yields a redundant clause with respect to $N$ or is contained in $N$.

**Subsumption** $(N \uplus \{C_1, C_2\}) \Rightarrow_{\text{SUP}} (N \cup \{C_1\})$

provided $C_1 \subset C_2$

**Tautology Deletion** $(N \uplus \{C \vee P(t_1, \ldots, t_n) \vee \neg P(t_1, \ldots, t_n)\})$
$\Rightarrow_{\text{SUP}} (N)$

**Condensation** $(N \uplus \{C_1 \vee L \vee L\}) \Rightarrow_{\text{SUP}} (N \cup \{C_1 \vee L\})$

**Subsumption Resolution** $(N \uplus \{C_1 \vee L, C_2 \vee \neg L\}) \Rightarrow_{\text{SUP}}$
$(N \cup \{C_1 \vee L, C_2\})$

where $C_1 \subseteq C_2$